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Finite-size corrections to Lyapunov spectra for band random matrices

T Kottos†, A Politi‡ and F M Izrailev§

† Department of Physics of Complex Systems, The Weizmann Institute of Science, Rehovot 76100, Israel, and Department of Physics, University of Crete and Research Centre of Crete, PO Box 1527, 71110 Heraklion, Crete, Greece

‡ Istituto Nazionale di Ottica, 50125 Firenze, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Firenze, Italy

§ Instituto de Fisica, Universidad Autonoma de Puebla, Apartado Postal J-48, Colonial San Manuel, Puebla, 72570, Mexico, and Budker Institute of Nuclear Physics, Novosibirsk 630090, Russia

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Abstract. The transfer-matrix method is applied to quasi-one-dimensional and one-dimensional disordered systems with long-range interactions described by band random matrices. We investigate the convergence properties of the entire Lyapunov spectra of finite samples as a function of the bandwidth and of the sample length. Different scaling laws are found with respect to what is suggested by the analysis of the localization properties of the eigenfunctions. Our results, at variance with the Anderson model, suggest that the contacts of a finite sample with the leads play a prominent role.

1. Introduction

Lyapunov exponents represent one of the main tools in the study of both disordered and dynamical systems. In the former case, they arise from the application of the transfer-matrix method and allow one determine, e.g., the localization length, which corresponds to the inverse of the minimum positive Lyapunov exponent. In the latter case, starting from the evolution in the tangent space, one is interested in determining, among other quantities, the Kolmogorov–Sinai entropy, which can be expressed as the sum of the positive exponents.

Recently, growing attention has been given to the study of high-dimensional systems such as coupled maps, chains of nonlinear oscillators, dynamical models with delayed feedback, disordered systems in two and three dimensions, and one-dimensional models with long-range interaction. In all of these cases, the so-called Lyapunov spectrum is defined as the sequence of the Lyapunov exponents γ_i (ordered according to increasing/decreasing size) represented as a function of i/D, where D is the total number of exponents. Many numerical simulations and analytical arguments indicate the existence of a limit spectrum for $D \to \infty$ (see, e.g., [1] and references therein). Compared to the case for the largest and smallest exponents, for which some rigorous mathematical results have been obtained, the properties of the 'bulk' of the Lyapunov spectrum are less well understood.

In reference [2], the scaling properties of the Lyapunov spectrum were studied for disordered systems described by infinite band random matrices (BRM). Such matrices have been extensively investigated in connection with one-dimensional Anderson-type models

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with long-range random hopping, as well as with quasi-one-dimensional thin wires (see, e.g., [3] and references therein). In particular, the scaling properties of the eigenfunction localization lengths proved to be in accordance with the predictions [4, 5] based on results for dynamical quantum models that are strongly chaotic in the classical limit.

Since any disordered sample used in practical applications is finite, it is useful to study not only the asymptotic value of the Lyapunov exponents but also the so-called effective exponents, i.e. the exponents actually observed for samples of finite length N. They represent a useful tool for quantifying the effect of the coupling with the leads, which is definitely one of the major ingredients contributing to finite-size corrections. Moreover, information about the statistical properties of effective Lyapunov exponents can shed further light on, e.g., the fluctuations of the conductance in the metallic regime [6–10].

In this paper, we investigate the convergence properties of the effective Lyapunov exponents of finite BRMs associated with finite samples embedded in an otherwise perfectly ordered lattice. Our investigation suggests that different parts of the spectrum exhibit different convergence properties. This is particularly clear from our data for the maximal Lyapunov exponent as compared to the bulk of the spectrum.

The outline of the paper is as follows. In section 2 the connection between the Hamiltonian band matrix model and the conductance of disordered samples is summarized with the purpose of introducing the appropriate Lyapunov exponents. In section 3, we recall some known results from scaling theory for similar problems: they will represent the starting point for the numerical investigation carried on in the following section, section 4. Some final comments and conclusions are presented in the last section, section 5.

2. The model

The general model describing quasi-one-dimensional or one-dimensional systems with long-range hopping is given by the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}c_n(t)}{\mathrm{d}t} = \sum_{m=n-b}^{n+b} H_{n,m}c_m(t) \tag{1}$$

where $c_n(t)$ is the probability amplitude for an electron to be at site n and $H_{n,m}$ is a symmetric band random matrix. Specifically, the entries of $H_{n,m}$ are independent Gaussian variables with zero mean and variance $\sigma^2 = 1 + \delta_{n,m}$ for $|n - m| \leqslant b$, while the matrix elements outside the band are all set equal to zero. In one-dimensional geometry, the parameter b defines the range of hopping between neighbouring sites, while in the quasi-one-dimensional interpretation, this parameter has the meaning of the number of transverse channels for the scattering waves along a thin wire [3].

The insertion of a disordered sample of length N into a perfectly ordered lattice requires the definition of two proper leads at the extrema of the sample. At variance with the standard Anderson model, where only nearest-neighbour couplings are present, our model has long-range hopping terms that allow some freedom in the choice of the structure of the leads. A reasonable choice consists in assuming a band structure in the ordered lattice with the same width b, and the hopping elements $H_{n,m}$ all set equal to U (for the sake of simplicity we choose $U = 1$), while the hopping amplitudes coupling the leads with sample sites are randomly chosen with the same distribution as in the core of the sample (the intermediate regions connecting the samples with the leads will hereafter be called 'contacts': they extend over b sites). As an example, we show below the Hamiltonian structure for $b = 2$ (asterisks

indicate random elements):

$$\begin{array}{ccccccccccc}
\ldots & \ldots & 1 & 1 & & & & & & & \\
1 & 1 & 1 & 1 & * & & & & & & \\
 & 1 & 1 & 1 & * & * & & & & & \\
 & & * & * & * & * & \ldots & & & & \\
 & & & & & \ldots & * & * & * & * & \\
 & & & & & & * & * & 1 & 1 & 1 \\
 & & & & & & & * & 1 & 1 & 1 \quad \ldots .
\end{array}$$

Accordingly, a finite sample of length N consists in practice of an entirely disordered bulk of length $N - 2b$ and two partially disordered contacts each of length b.

The eigenvalue equation is obtained by inserting the standard *ansatz* $c_n(t) = \exp(-\mathrm{i}Et)\psi_n$ in equation (1). The resulting equation can be cast in the form of a $2b$-dimensional linear map along the spatial direction:

$$\mathbf{\Psi}(n+1) = \mathbf{T}(n)\mathbf{\Psi}(n) \tag{2}$$

where $\Psi_i(n) \equiv \psi_{n+b-i}$ and the matrix $\mathbf{T}(n)$ is defined as follows:

$$\begin{aligned}
&[\mathbf{T}(n)]_{1,j} = \frac{1}{H_{n,n+b}}(\delta_{j,b}E - H_{n,n+b-j}) \\
&[\mathbf{T}(n)]_{i,j} = \delta_{i-1,j} \qquad [\mathbf{T}(n)]_{i,2b} = 0 \qquad 2 \leqslant i \leqslant 2b.
\end{aligned} \tag{3}$$

In this picture, an eigenstate of equation (1) can be treated as a 'trajectory' of the random map (3), and its localization properties are determined by the Lyapunov exponents.

In a previous paper [2] we investigated the shape of the Lyapunov spectrum in the limit of infinitely extended disordered samples. Here, since we are interested in finite samples of size N, one should introduce the transfer matrix $\mathbf{T} = \prod_{n=1}^{N}\mathbf{T}(n)$, which couples two opposite leads. As was shown in [6], the matrix $\mathbf{T}$ satisfies the following property, not shared by the single matrix $\mathbf{T}(n)$:

$$\mathbf{T}^{\mathrm{t}}\mathbf{\Sigma}\mathbf{T} = \mathbf{\Sigma} \qquad \mathbf{\Sigma} = \begin{pmatrix} 0 & \mathbf{S} \\ -\mathbf{S}^{\mathrm{t}} & 0 \end{pmatrix} \tag{4}$$

where $\mathbf{S}$ is a lower triangular matrix of size b, with $S_{ij} = 1$, $i \geqslant j$. In fact, this property corresponds to the flux conservation in the process of the scattering of a wave through the sample.

It is convenient to describe the scattering states in terms of 'eigenfunctions' of the free dynamics occurring in the ordered region. The eigenvalues of the corresponding Hamiltonian, defined by setting all random elements equal to 1, are

$$E(p) = 1 + 2\cos p + \cdots + 2\cos(bp) = \frac{\sin[(2b+1)(p/2)]}{\sin(p/2)} \qquad p \in (-\pi, \pi) \tag{5}$$

while the corresponding eigenvectors are

$$\psi_n(p) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\pm \mathrm{i}np}. \tag{6}$$

For any fixed energy value $\tilde{E}$, there are $\nu \leqslant b$ pairs of opposite real solutions of the equation $E(p_k) = \tilde{E}$. Each pair corresponds to an open channel, or propagation mode, sustaining waves with opposite velocities. In this paper, we limit ourselves to studying the case where $\tilde{E} = 0$, in which all channels are open, i.e. $\nu = b$, and the admissible momenta are equally

spaced. In what follows we shall assume that the momenta p_k are ordered in such a way that positive velocities correspond to the first b elements:

$$p_1, p_2, \ldots, p_b, -p_1, \ldots, -p_b \qquad p_k = (-1)^k \frac{2\pi k}{2b+1} \qquad k = 1, \ldots, b. \tag{7}$$

In fact, one can see that the corresponding velocities $v(p) = \mathrm{d}E/\mathrm{d}p$ for $E = 0$ are given by

$$v_k = 2b + 1 \Big/ \left(2 \sin\left[\frac{\pi k}{2b+1} \right] \right) \qquad v_{k+b} = -v_k \qquad k = 1, \ldots, b. \tag{8}$$

Instead of defining the initial state in the scattering process in terms of the probability amplitude at $2b$ consecutive sites (as required by the standard representation of the vector $\boldsymbol{\Psi}(n)$), one can refer to the $2b$ amplitudes of the plane waves sustained by the ordered lattice. It can easily be checked that the transformation from the momentum to the position representation is defined by the matrix

$$[\mathbf{U}(n)]_{j,k} = \exp(\mathrm{i}(j+n)p_k) \tag{9}$$

and hence that scattering matrix can be written as

$$\mathbf{M} = \mathbf{Z}^{-N}\mathbf{U}(0)^{-1}\mathbf{T}\mathbf{U}(0) \tag{10}$$

where $\mathbf{Z}$ is a diagonal matrix whose entries, $Z_{j,j} = \mathrm{e}^{\mathrm{i}p_j}$, account for the phase difference between the sites $n = 1$ and $n = N$. The matrix $\mathbf{M}$ has an almost symplectic structure; indeed, it satisfies the relation $\mathbf{M}^\dagger\mathbf{V}\mathbf{M} = \mathbf{V}$, where $\mathbf{V}$ is a diagonal matrix with $V_{j,j} = v_j$ of the same order as before [6].

For the determination of the conductance, one needs to introduce the matrix $\mathbf{F}$ connecting flux amplitudes:

$$\mathbf{F} = \boldsymbol{\Gamma}\mathbf{M}\boldsymbol{\Gamma}^{-1} \tag{11}$$

where the diagonal $2b \times 2b$ matrix $\boldsymbol{\Gamma}$ is defined as $\Gamma_{i,j} = \delta_{i,j}\sqrt{|v_i|}$. The above transformation is equivalent to the normalization of the scattering matrix, and it takes into account the fact that the waves propagate with different velocities in different open channels.

The transfer matrix $\mathbf{F}$ connecting the incoming with the outgoing flux amplitudes in the various channels turns out to be symplectic, as it satisfies the relation

$$\mathbf{F}^\dagger\boldsymbol{\sigma}_3\mathbf{F} = \boldsymbol{\sigma}_3 \tag{12}$$

where $\boldsymbol{\sigma}_3$ is the generalized Pauli matrix:

$$\boldsymbol{\sigma}_3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \tag{13}$$

and $\mathbf{1}$ denotes a $b \times b$ identity matrix. Notice that condition (12) follows essentially from the flux conservation, i.e. from the unitarity of the scattering matrix.

The matrix $\mathbf{F}$ is the key ingredient for the determination of the conductivity from the Landauer formula (see, e.g., [7] for a general derivation and [10] for an application in the specific case of BRMs). More precisely, it is necessary to compute the Lyapunov exponents

$$\gamma_i(N) = \frac{1}{2N} \ln \lambda_i(N) \tag{14}$$

where $\lambda_i(N)$ denote the (real) eigenvalues of the matrix $\mathbf{F}^\dagger\mathbf{F}$. The Lyapunov exponents will be conventionally ordered from the maximum to the minimum one; that is, $\gamma_1(N) > \gamma_2(N) > \cdots > \gamma_i(N) > \cdots > \gamma_{2b}(N)$. Because of the symplectic structure of $\mathbf{F}$, the

exponents are arranged in b pairs with opposite values ($\gamma_i(\nu) = -\gamma_{2b-i+1}(\nu)$); for this reason, in the following, we will always report only the positive exponents (i.e. $i \leqslant b$).

In the limit of infinitely long samples ($N \to \infty$), the ergodic multiplicative theorem [11] ensures that the statistical fluctuations of the effective Lyapunov exponents vanish: in fact, the quantities $\gamma_i(\infty)$ are, by definition, the Lyapunov exponents of the infinite product of matrices composing **F**. Moreover, in the limit $N \to \infty$, the effect of the similarity transformations involved in the definition (10), (11) of **F** becomes negligible, so the values of $\gamma_i(N)$ converge also to the Lyapunov exponents of the matrix **T**. However, as long as one deals with finite samples, the effective Lyapunov exponents $\gamma_i(N)$ depend on the realization of the disorder. It is therefore convenient to average $\gamma_i(N)$ over the ensemble of all possible realizations. In order not to overload the notation, this average will always be understood.

3. Scaling behaviour

Let us first discuss the scaling behaviour of the Lyapunov exponents for the Anderson model. In fact, the transfer-matrix approach reveals a clear analogy between the physical problem considered in the present paper and the Anderson localization in a stripe. Indeed, the bandwidth b plays the same role as the strip width L_t in the sense that both define the number of possible channels for electronic conductance. It is important to note, however, that for the analogy to be kept as strict as possible, one must assume that the Lyapunov exponents are measured in units of the interaction range (or, equivalently, in mean free paths), i.e. by referring to the lattice spacing in the Anderson problem and to b in the present case. This feature was already noticed in [2], where it was pointed out that the Lyapunov spectra of BRM, measured in natural units, scale as $1/b$.

One cannot straightforwardly apply to the present case the single-channel scaling theory in order to infer localization properties for different disorder amplitudes and 'transverse widths', since there is no proper localization length in the thermodynamic limit $b \to \infty$. In fact, while it is conjectured that the minimum Lyapunov exponent is finite in the limit of infinitely large stripes ($L_t \to \infty$) in the Anderson problem (in the insulating regime), it vanishes as $1/b$ for band random matrices, even using the appropriate spatial scale. Indeed, let us recall that the localization length for the eigenfunctions of energy E is $l_\infty(E) = (2b^2/3)[1 - (E^2/(8b))]$ [3], i.e. it diverges as b^2 over the whole energy range.

Another feature of the scaling behaviour that has been investigated in the Anderson problem concerns the dependence of the Lyapunov exponents on the sample length for fixed transverse width L_t. In reference [9] it was found that the scale dependence for the Anderson model has the form

$$\gamma_j(N) = \gamma_j(\infty) F_j(\gamma_j(\infty) N) \tag{15}$$

independently of the disorder amplitude. In the context of BRMs, the above relation is somewhat trivial, since the amplitude of the disorder can be scaled out. This is immediately realized by noticing that the elements of the transfer matrices involve only ratios of the disorder terms (apart from the energy term which is the only contribution that needs to be appropriately rescaled), thus revealing that their absolute scale is irrelevant.

Meanwhile, a different approach has been suggested for describing scaling properties in BRMs without leads. Such a procedure involves the introduction of the generalized localization length $l_q(N)$ [5] of a generic eigenvector:

$$l_q(N) = \exp\langle \mathcal{H}_q \rangle \tag{16}$$

where

$$\mathcal{H}_1 = -\sum_n |\psi_n|^2 \ln(|\psi_n|^2)$$
$$\mathcal{H}_q = \frac{1}{1-q} \ln P_q \qquad P_q = \sum_{n=1}^{N} |\psi_n|^{2q} \qquad q \geqslant 2 \tag{17}$$

and ψ_n is the nth component of an eigenvector of the matrix. The averaging of $\mathcal{H}_q$ in equation (16) is performed over the disorder and over the eigenstates corresponding to energies within a prespecified window. It was numerically shown in [5] and analytically proved in [3] that the rescaled localization length $l_q(N)/l_q^{GOE}(N)$, where l_q^{GOE} corresponds to full random matrices, depends only on the ratio l_∞/N. More detailed analytical studies [3] have revealed that the scaling behaviour for $l_q(N)$ is very close to the form

$$l_q^{-1}(N) = l_q^{-1}(\infty) + C_q/N \tag{18}$$

for $q \neq 2$, while it holds exactly for $q = 2$. Notice that, in the latter case, the localization length $l_2(N)$ is related to the inverse participation ratio which has the simple physical meaning of the probability for a quantum particle to return to the initial position after infinite time. The second term in the r.h.s. of expression (18) represents the normalization factor $l_q^{GOE}(N)$: it was found both numerically and analytically that the coefficient C_q is always positive and independent of N. The positiveness of C_q indicates that the finite-length estimates of the localization length converge to its asymptotic value $l_q(\infty)$ from above.

Let us finally mention that the scaling relation, equation (18), appears to be rather general, as revealed by numerical simulations performed for many other models, like the kicked rotator [4, 12], the one-dimensional Anderson and Lloyd models [13], and one-dimensional dimer models [14, 15].

4. Numerical analysis

4.1. The method

As is mentioned in section 2, the conductance of a finite sample can be determined from the eigenvalues of $\mathbf{F}^\dagger\mathbf{F}$. For this reason, here we study the convergence properties of the latter quantities by varying the sample size N and the bandwidth b. The standard technique for the determination of the eigenvalues runs into trouble even for relatively short samples because of numerical inaccuracies due to the small denominators in equation (3). Such a difficulty can be overcome by suitably modifying the standard algorithm for the computation of the Lyapunov exponents of an infinite product of matrices [16]. In fact, given a finite sample of length N (and the corresponding matrix $\mathbf{F}$ defined as in equation (11)), one can formally construct the following infinite product of matrices:

$$\cdots \mathbf{F}^\dagger\mathbf{F} \cdots \mathbf{F}^\dagger\mathbf{F} \cdots \mathbf{F}^\dagger\mathbf{F} \cdots . \tag{19}$$

Such a sequence can be recursively applied to b independent vectors, orthonormalizing them every single step. Accordingly, one finds b Lyapunov exponents that are nothing but the logarithms of the (real) eigenvalues of $\mathbf{F}^\dagger\mathbf{F}$. The advantage of this procedure over the standard diagonalization methods is that the orthonormalization can be implemented for all intermediate steps in the construction of $\mathbf{F}$ (i.e. multiplication by the transfer matrices $\mathbf{T}$ and application of the similarity transformations). This approach has already proved its effectiveness in the study of the Anderson problem [17].

In practice, the number $\mathcal{M}$ of 'replicas' of $\mathbf{F}^\dagger\mathbf{F}$ in (19) is finite: we have chosen $\mathcal{M}$ so as to guarantee at least an accuracy 10^{-4} for all of the Lyapunov exponents, i.e. $\mathcal{M} > 2500$.

4.2. Convergence to the asymptotic limit

The main goal of this subsection is to study the convergence of $\gamma_i(b, N)$ towards the set of self-averaged Lyapunov exponents $\gamma_i(b, N \to \infty)$ as a function of the sample length N and of the bandwidth b.

We have already seen that BRMs have somewhat peculiar properties which make the application of the standard scaling theory to Lyapunov exponents problematic. One problem which, however, makes perfect sense in the context of BRMs, is the identification of a single relationship expressing the dependence of the Lyapunov exponents on both the sample length N and the 'transverse' width b. This is an issue that has not yet received a clear exposition in the two- and three-dimensional Anderson problems [9].

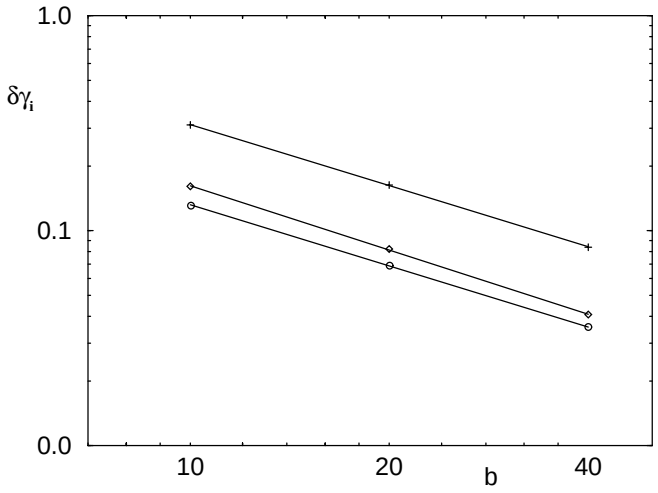

Figure 1. The convergence of the Lyapunov exponents $\gamma_i(b, \infty)$ is investigated by plotting the difference $\delta\gamma_i = \gamma_i(b, \infty) - \gamma_i(\infty, \infty)$ versus b. The three data sets refer to $i/b = 0.3$ (plus signs), 0.5 (diamonds), and 0.7 (circles). The straight lines follow from a best fit: their slope is close to -1 in all cases (with deviations of a few per cent).

The natural starting point is represented by equation (18), which gives the localization length as measured directly from the eigenfunctions of the Hamiltonian. On recalling the b^2-dependence of $l_q(\infty)$, one realizes that the finite-size corrections depend only on a scaling parameter, namely N/b^2. Accordingly, one could expect the appropriate scaling relation for the Lyapunov exponents to be of the type

$$\gamma_i(b, N)/\gamma_i(b, \infty) = f(i/b, N/b^2) \tag{20}$$

where we have added a dependence on i/b to account for possible differences exhibited by the various exponents. However, a careful analysis of our data definitely rules out such a possibility, so one needs to modify the above *ansatz* in a more suitable manner. After many different attempts to find the correct scaling dependence on b and N, we have come to the conclusion that the most convincing and yet simple scaling hypothesis is that

$$\gamma_i(b, N)/\gamma_i(b, \infty) = f(\gamma_i(b, \infty)Nb^{\alpha_i}) \tag{21}$$

with α_i some function of the ratio i/b.

First, it is necessary to determine the asymptotic Lyapunov exponents ($N \to \infty$) for different bandwidths. In this limit, the 'contacts' between the ordered regions and the sample

play no role, like the similarity transformations involved in the definition of **F**. Accordingly, one can get rid of most of the technical difficulties and determine $\gamma_i(b,\infty)$ by resorting to the usual transfer-matrix approach as implemented in reference [2]. The results reported in figure 1 indicate a convergence of the type $1/b$ for the bulk of the spectrum.

An effective test of scaling relation (21) can be made after subtracting the asymptotic value 1 from both sides, i.e. by studying the behaviour of the difference

$$\delta\gamma_i(b,N) \equiv 1 - \gamma_i(b,N)/\gamma_i(b,\infty). \tag{22}$$

With the asymptotic values of γ_i determined, we have plotted the finite-size correction $\delta\gamma_i$ versus the rescaled sample length $m = \gamma_i(b,\infty)Nb_i^{\alpha}$ for different choices of α_i. In all cases, we find that $\delta\gamma_i$ is positive, indicating that the convergence to the asymptotic values is from below. This striking difference from the behaviour of the directly computed localization length (see equation (18)) is the clearest indication that the influence of the leads and the type of the contacts results in strong finite-size corrections.

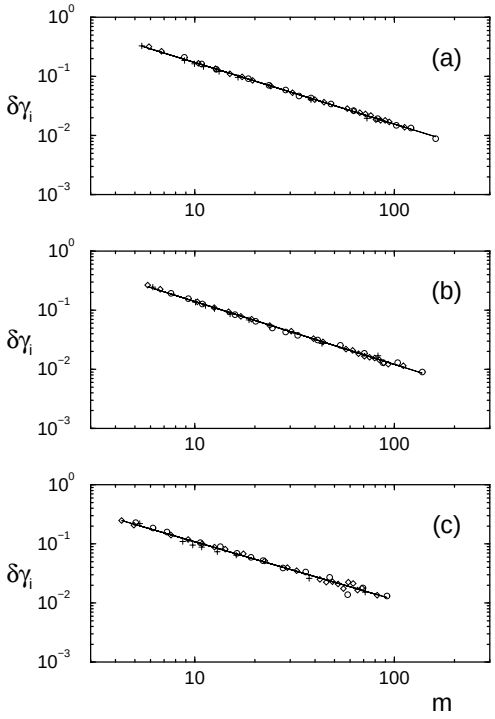

Figure 2. Rescaled finite-size corrections $\delta\gamma_i$ to the Lyapunov exponents versus the rescaled length of the sample size m: circles correspond to $b = 10$, diamonds to $b = 20$, and plus signs to $b = 40$; (a), (b), and (c) correspond to $i/b = 0.3$, 0.5, and 0.9, respectively. The straight lines are the best fits. The deviations of the resulting slope from -1 are in all cases approximately 3%.

The best data collapses obtained for $i/b = 0.3$, 0.5, and 0.9 are reported in figure 2. In all cases, $f(m)$ turns out to be essentially equal to $1 - A/m$ with the value of A depending very little on i/b ($A \approx 1.1$). As anticipated in equation (21), the main dependence on i/b is contained in the exponent α_i which appears to change linearly with i:

$$\alpha_i = i/b - 0.3. \tag{23}$$

Notice that the reason for replacing the initial *ansatz* with equation (21) is contained fully in the above expression for α_i. In fact, equation (20) is consistent with equation (21) only if $\alpha_i = -1$ for $0 < i/b < 1$, which is not the case. A partial explanation for this result comes from the observation that the bulk of the Lyapunov spectra scales in a different way from the minimum value (to which equation (20) refers). However, this does not fully explain why the various parts of the Lyapunov spectrum exhibit different convergence properties.

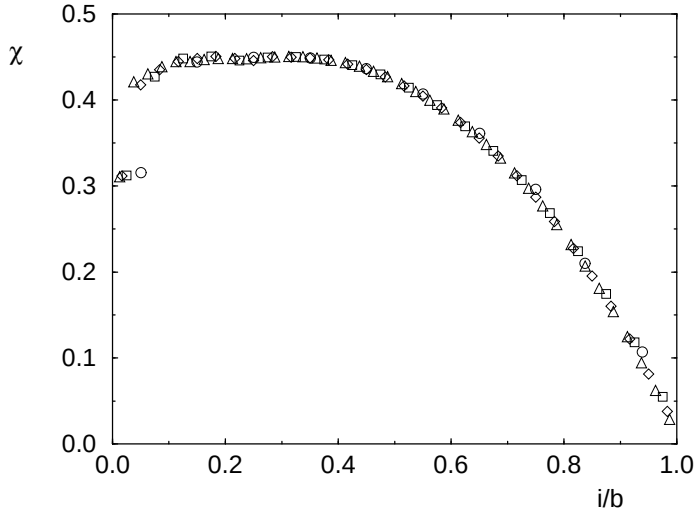

Figure 3. The Lyapunov spectrum as determined for $b = 10$ (circles), 20 (squares), 30 (diamonds), and 40 (triangles).

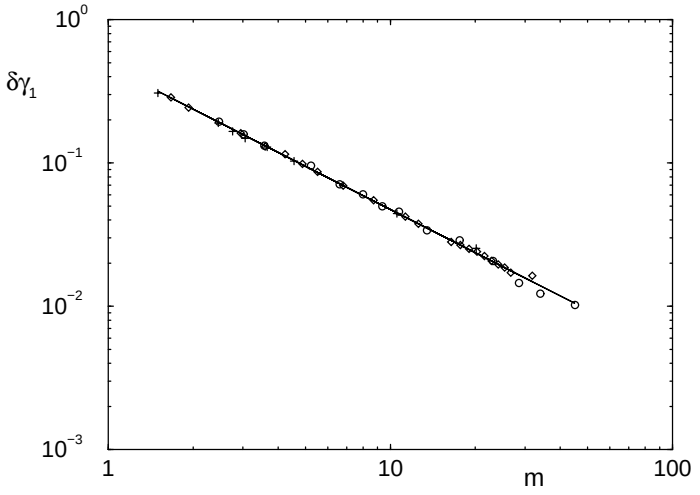

Figure 4. Finite-size corrections to the maximum Lyapunov exponent, using the same representation as in figure 2. The slope of the straight line is -1.

A clear counter-example to relationship (23) is found by analysing the behaviour of the maximal exponent. This is not a surprise, since in reference [2] the existence of a 'phase

transition' in the Lyapunov spectrum occurring approximately at $i/b = 0.1$ had already been noticed. This is illustrated in figure 3, where $\chi = \gamma_i i$ is plotted versus i/b, revealing an incipient discontinuity in the derivative of the spectrum. It is thus reasonable that different convergence properties are observed above and below $i/b \approx 0.1$. In fact, we find that the convergence is of the type $1/N$ in both cases, but that the value of α_1 is -1 for the maximum exponent, as seen in figure 4 (it should be recalled that γ_1 exhibits a different scaling behaviour from that in the bulk of the spectrum, being independent of b).

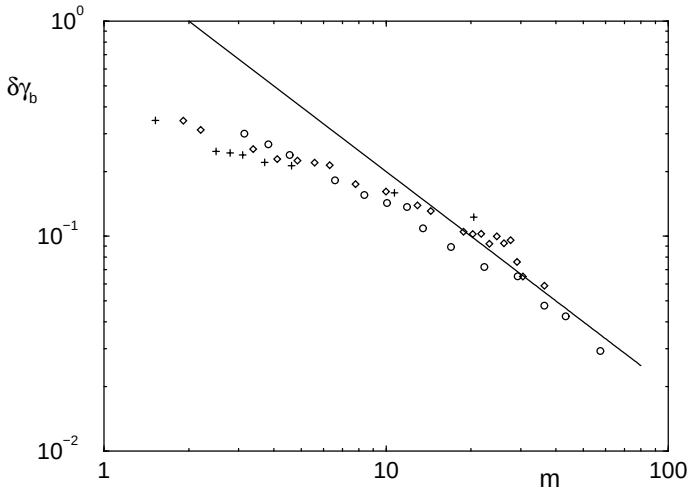

Figure 5. Finite-size corrections to the minimum Lyapunov exponent, using the same representation as in figure 2. The straight line with slope -1 is drawn for reference.

The behaviour of the minimal exponent is another important test, but before discussing this case, we would like to stress that the scaling of γ_b as $1/b^2$ makes the numerical computations much more difficult: in fact, it is very hard to get rid of statistical fluctuations when b becomes relatively large. An indication of this difficulty is already given by the comparison of the best overlaps obtained in the various cases: increasing fluctuations are detected upon increasing i/b, testifying to the importance of statistical fluctuations (this is particularly visible in figure 2(c), i.e. for $i/b = 0.9$). Nevertheless, one can see in figure 5 that the data collapse is still not bad on assuming that $\alpha_b = 0.7$, i.e. the value predicted by the linear law (23). Consistency with equation (20) would require $\alpha_b = 0$, which definitely gives a much worse overlap of the various curves and has to be, therefore, ruled out. Moreover, a regression of the various curves corresponding to different values of b seems to suggest that the convergence to the asymptotic value in this last case is slower than $1/N$, but it is not clear whether this is an artifact due to a lack of sufficient statistics or whether it is a finite-size (N) effect.

After having presented a possible unified description of the convergence properties of 'finite-length' Lyapunov exponents, it is worth discussing the origin of the corrections expressed by f in equation (21). One reason for these corrections is the presence of the 'contacts'. Since the 'contacts' are less disordered than the bulk, one can expect them to be characterized by a different, smaller, growth rate. The first prediction of this argument is a negative sign for the correction, i.e. a convergence from below, as is indeed observed. This accounts for the main difference of the localization properties. Moreover, if this were the only source of corrections, and if there were no boundary effects between the leads

and the sample, one should conclude that the relative correction must be proportional to b/N, i.e. proportional to the ratio between the length of the leads and the length of the sample. This statement is equivalent to saying that equation (21) holds with $\alpha_1 = -1$ for the maximal Lyapunov exponent. As we have already seen, this prediction is fully confirmed. Furthermore, direct numerical simulations made to compute the growth rate in the contacts and that in the sample separately do reveal that the former contribution is half of the latter, almost independently of b. This proves directly the correctness of our simple conjecture.

The same argument, applied to the rest of the spectrum, would imply that the correction is still of the order of b/N, which means that the α_i-value in equation (21) is zero independently of i/b (except for the minimum). Since a strictly positive α_i is found, instead, for $i/b > 0.3$, this means that the actual correction is smaller than expected from the above argument. We can only give a qualitative explanation for the discrepancy: as long as i/b is strictly larger than $1/b$ (in the asymptotic limit $b \to \infty$), the corresponding growth rate γ_i is of the order of $1/b$ (except for in the limit case of the minimum exponent), so the contribution to the expansion observed in the 'contacts' is not uncoupled from the expansion in the rest of the sample, and this makes the subdivision of the entire sample into a bulk and two 'contacts' less well defined. Moreover, we should add that, even in the absence of the leads, finite-size corrections must be present and, as far as we are aware, there are no theoretical predictions about this kind of correction apart from those for the maximal exponent [18].

5. Conclusions and a forward look

We have studied finite-length Lyapunov spectra of symmetric band random matrices describing quasi-one-dimensional and one-dimensional disordered systems with long-range interactions. Our main goal was to investigate the scaling properties of Lyapunov spectra upon changing both the bandwidth b and the sample size N. To our knowledge, the only example in the literature where a similar question has been addressed is reference [9], where the authors have commented on the global scaling behaviour with sample length and strip width. However, their conjectures are not supported by numerical analysis. Here, instead, detailed numerical investigations have led us to conjecture that

$$\gamma_i(b, N) \approx \gamma_i(b, \infty) - A\frac{b^{0.3-(i/b)}}{N} \tag{24}$$

for $i/b \gtrsim 0.1$. In fact, it was already clear in reference [2] that for $i \ll b$ the Lyapunov spectrum exhibits different scaling properties, and, therefore, it is no surprise that the convergence to the asymptotic shape is also different in the same region. Anyway, the most important application of the above result is in connection with the conductance in the localized and metallic regimes. In both regimes, it is the lowest Lyapunov exponents with $i \approx b$ that play a major role. Let us, therefore, consider the minimum exponent ($i = b$). In this case, in the limit b and $N \to \infty$ but for finite b^2/N (the latter is the standard scaling parameter used to discriminate between metallic and localized regimes), one can notice that expression (24) implies that the correction term is asymptotically negligible with respect to the leading term $\gamma_i(b, \infty)$. The implications of this result for the behaviour of the conductance are thoroughly analysed in [10].

Even more striking is—at variance with the prediction of the two-dimensional Anderson model—the negative sign of the finite-size correction terms. This result and, more generally, the global scaling behaviour can be partly attributed to the influence of the 'contacts'

connecting the ordered leads with the disordered sample. However, while the dependence on b/N for the maximal exponent is also supported by a simple theoretical argument, the same cannot be said for the other exponents. Nevertheless, we wish to recall that other combinations of simple functions provide definitely less accurate descriptions of the observed data. On the other hand, we cannot exclude the possibility that the relative 'smallness' of the values of b and N accessible to a numerical analysis masks a dependence different from that conjectured in equation (21). We can only stress that the scaling *ansatz* is chosen as by far the simplest one among a set of otherwise even less convincing alternatives. In other words, our work represents an instance of the application of Occam's razor. Thus, some theoretical progress is needed to shed further light on this problem. This is particularly true for the minimal exponent, which requires the most delicacy to determine numerically.

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References

[1] Lepri S, Politi A and Torcini A 1996 *J. Stat. Phys.* **82** 1429
[2] Kottos T, Politi A, Izrailev F M and Ruffo S 1996 *Phys. Rev.* E **53** R5553
[3] Fyodorov Y F and Mirlin A D 1994 *Int. J. Mod. Phys.* **8** 3795
[4] Casati G, Guarneri I, Izrailev F M and Scharf R 1990 *Phys. Rev. Lett.* **64** 5
[5] Casati G, Molinari L and Izrailev F M 1990 *Phys. Rev. Lett.* **64** 16
Evangelou S N and Economou E N 1991 *Phys. Lett.* **151A** 345
[6] Molinari L 1997 *J. Phys. A: Math. Gen.* **30** 983
[7] Pichard J-L 1984 *Thesis* No 2858, University of Paris, Orsay
[8] Imry Y 1986 *Europhys. Lett.* **1** 249
[9] Pichard J-L, Zanon N, Imry Y and Stone A D 1990 *J. Physique* **51** 587
Pichard J-L 1986 *J. Phys. C: Solid State Phys.* **19** 1519
[10] Kottos T, Izrailev F M and Politi A 1998 *Physica* D at press
[11] Oseledec V I 1968 *Trans. Moscow Math. Soc.* **19** 197
[12] Izrailev F M 1990 *Phys. Rep.* **196** 299
[13] Casati G, Fishman S, Guarneri I, Izrailev F M and Molinari L 1992 *J. Phys.: Condens. Matter* **4** 149
[14] Izrailev F M, Kottos T and Tsironis G P 1996 *J. Phys.: Condens. Matter* **8** 2823
[15] Varga I and Pipek J 1997 *Preprint* cond-mat/9710247
[16] Benettin G, Galgani L, Giorgilli A and Strelcyn J M 1980 *Meccanica* **15** 9
Benettin G, Galgani L, Giorgilli A and Strelcyn J M 1980 *Meccanica* **15** 21
[17] Markos P and Kramer B 1993 *Phil. Mag.* B **68** 357
Markos P and Kramer B 1993 *Ann. Phys., Lpz.* **2** 339
[18] Pikovsky A and Politi A 1998 *Nonlinearity* **11** 1049